Formal Theory of Landau Damping

→ Formal Theory of Landau Damping

Consider initial value problem:

$$f(t=0) = \langle f(v) \rangle + \tilde{f}(0, v, x)$$

Evolution of ϕ ?

(i.) Landau Solution

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{q}{m} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\nabla^2 \tilde{\phi} = -4\pi n_0 q \int \tilde{f} \, dv$$

$$\frac{\partial \tilde{f}_k}{\partial t} + ikv \tilde{f}_k = ik \tilde{\phi}_k \frac{q}{m} \frac{\partial \langle f \rangle}{\partial v}$$

$$k^2 \tilde{\phi}_k = 4\pi n_0 q \int \tilde{f}_k \, dv$$

Laplace Transform: $\displaystyle \phi_{k,\omega} = \int_0^\infty e^{i\omega t} \phi_k(t)$

$\mathrm{Im}\ \omega > 0$

$$\phi_k(t) = \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} e^{-i\omega t} \phi_{k,\omega} \frac{d\omega}{2\pi}$$

then: $\displaystyle\int_0^\infty e^{i\omega t}\frac{\partial \tilde{f}_k}{\partial t} = -\tilde{f}_k(v,0) - i\omega \int_0^\infty e^{i\omega t}\tilde{f}_k$

$$= -\tilde{f}_k(v,0) - i\omega\, \tilde{f}_{k,\omega}$$

$$-\tilde{f}_k(v,0) - i(\omega - kv)\tilde{f}_{k,\omega} = i\frac{q}{m}k\,\tilde{\phi}_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}$$

$$\tilde{f}_{k,\omega} = i\frac{\tilde{f}_k(v,0)}{(\omega - kv)} - \frac{q}{m}\frac{k}{(\omega - kv)}\phi_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}$$

$\therefore$

$$k^2\phi_{k,\omega} = 4\pi n_0 q\int dv\left\{-\frac{q}{m}\frac{k}{\omega - kv}\partial\langle f\rangle/\partial v\,\phi_{k,\omega} + i\frac{\tilde{f}_k(v,0)}{\omega - kv}\right\}$$

$\Rightarrow$

$$\epsilon(k,\omega)\phi_{k,\omega} = \frac{4\pi n_0 q i}{k^2}\int dv\,\frac{\tilde{f}_k(v,0)}{\omega - kv}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv\,\frac{\partial\langle f\rangle/\partial v}{\omega - kv}$$

$$\therefore \quad \phi_{k,\omega} = \frac{4\pi n_0 q}{k^2 \epsilon(k,\omega)}\, i \int dv\, \frac{\tilde{f}_k(v,0)}{\omega - kv}$$

Then,

$$\phi_k(t) = \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} d\omega\, \frac{4\pi n_0 q}{k^2 \epsilon(k,\omega)} \left(i \int dv\, \frac{\tilde{f}_k(v,0)}{\omega - kv} \right) e^{-i\omega t}$$

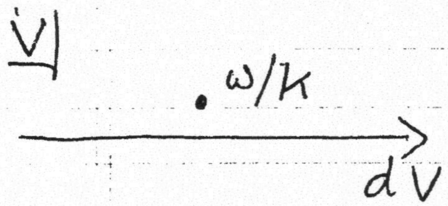
$\phi_k(t)$ determined by analytic structure of integrand

$\Rightarrow$ singularities $\displaystyle\int dv\, \frac{\tilde{f}_k(v,0)}{\omega - kv}$

$\Rightarrow \left\{ \begin{matrix} \text{zeroes} \\ \text{singularities} \end{matrix} \right\} \epsilon(k,\omega)$

Now: $\rightarrow \omega = \omega + i\epsilon \Rightarrow v = v - i\epsilon$

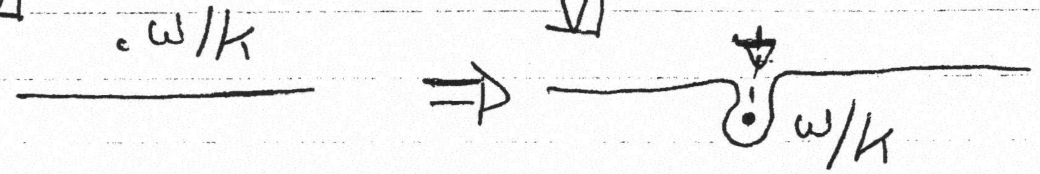
$\therefore$ so v integration along contour below pole at ω/k



If consider case of damped mode,

analytically continue by deforming contour so pole above v ct v

i.e.



$\therefore$

$\rightarrow$ singularities $\int dv\, \tilde{f}_k(v,0) \,/\, (\omega - kv) \Big|_{\text{analytic continuation}}$

only at singularities $\tilde{f}_k(v,0)$

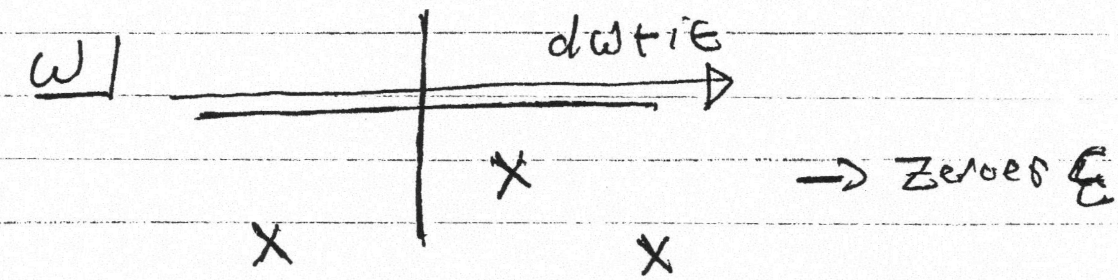
$\therefore$

$\rightarrow$ assuming $\tilde{f}_k(v,0)$ entire function
(no singularity of finite v) and normalizable

$\therefore$ $\displaystyle\int dv \frac{\tilde{f}_k(v,0)}{\omega - kv} \rightarrow$ entire function ω

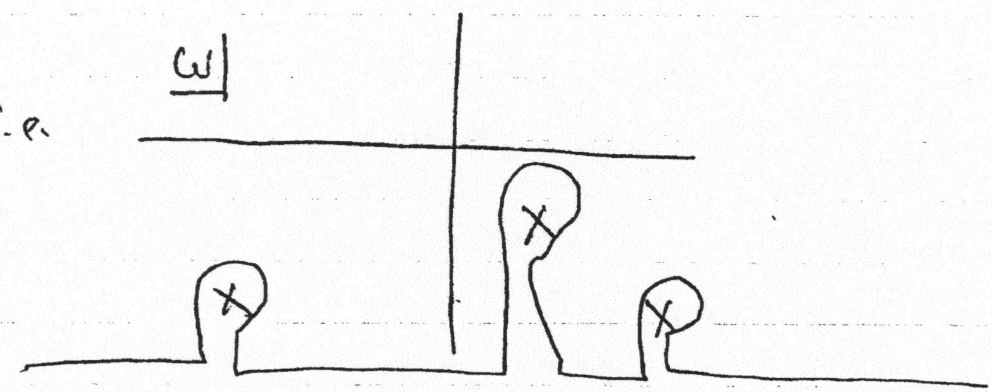
$\epsilon(k,\omega) \rightarrow$ entire function
(same argument)

$\therefore$ only singularities of integrand at zeroes $\epsilon(k,\omega)$



⇒ deform ω contour downward till encircles zeroes.

i.e.



Then;

$$\phi_k(t) = \sum_j \phi_k^j \, e^{-i\omega^j_{k,r}t} \, e^{-\omega^j_{k,Im}t}$$

↳ residue of j^{th} mode

So long time response dominated by least damped mode..

iii) Case - Van Kampen Solution (Schematic)

Aside: General solution of IVP

→ determine complete set of normal modes of system

→ evolution as normal modes with IVData + Normal Mode Evolution

i.e. Plucked string 

→ Fourier series with IVD ⇒ coefficients

→ Laplace Transform

For Vlasov Plasma ⟷ – Continuum of Singular Modes of f
– L.D. as phase mixing

For modes:

$$\frac{\partial \tilde{f}_k}{\partial t} + ikv\,\tilde{f}_k = i\frac{q}{m}k\tilde{\phi}_k\frac{\partial \langle f\rangle}{\partial v}$$

$$k^2\phi_k = 4\pi n_0 q\int \tilde{f}_k\,dv$$

$\therefore$

$$\frac{\partial f_k}{\partial t} + ikv f_k = i\frac{\omega_p^2}{k}\frac{\partial\langle f\rangle}{\partial V}\int dV\, f_k(V)$$

$$\Rightarrow \begin{cases} \dfrac{\partial f_k}{\partial t} + ikv f_k = -ik\,\eta(v)\displaystyle\int_{-\infty}^{+\infty} dv'\, f_k(v') \\ \eta(v) = -\dfrac{\omega_p^2}{k^2}\dfrac{\partial\langle f\rangle}{\partial V} \end{cases}$$

$$f_k = f_{k,\omega}\, e^{-i\omega t}$$

$$(V - \omega/k)\, f_{\omega/k}(v) = -\eta(v)\int_{-\infty}^{+\infty} dv'\, f_{\frac{\omega}{k}}(v')$$

$f = f(v, r)$

$r \equiv \omega/k$

$$(V - r)\, f_r(v) = -\eta(V)\int_{-\infty}^{+\infty} dv'\, f_r(v')$$

with normalization $\displaystyle\int_{-\infty}^{+\infty} dv\, f_r(v) = 1$

$$f_r(v) = -\frac{P\,\eta(v)}{V - r} + \lambda(r)\,\delta(V - r)$$

i.e. $(V-r)\,\delta(V-r) = 0$

$$1 = \int_{-\infty}^{+\infty} dv \left(-\frac{P\eta(v)}{v-r} + \lambda(r)\,\delta(v-r) \right)$$ Normalization

$\therefore$

$$\lambda(r) = 1 + \int_{-\infty}^{+\infty} dv \, \frac{P\eta(v)}{v-r}$$

so, normal modes f :

$\rightarrow$ $$f_r(v) = -\frac{P\eta(v)}{v-r} + \lambda(r)\,\delta(v-r)$$

$$\lambda(r) = 1 + \int_{-\infty}^{+\infty} dv \, \frac{P\eta(v)}{v-r}$$

$$\eta(v) = -\frac{\omega_p^2}{k^2} \frac{\partial \langle f \rangle}{\partial v}$$

$\rightarrow$ Modes $\begin{cases} \text{undamped} \\ \text{singular} \end{cases}$ $\Rightarrow$ correspond to ballistic modes (particle streams)

$\rightarrow$ Complete, Orthogonal Set (Case Ann. Phys. 7 349 1959)

Can super-pose to show equivalence to Landau solution; Damping via Phase-Mixing i.e. $\int e^{-v^2/v_T^2} e^{-ikvt} = \int dv\, e^{-\left(\frac{v}{v_T} + \frac{ikt v_T}{2}\right)^2} \cdot e^{-k^2 v_T^2 t^2/4}$

undamped ballistic mode

Mathematical Note:

$$\epsilon = 1 + \frac{\omega_p^2}{k} \int dv\, \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

$$= 1 - \frac{\omega_p^2}{k v_{Th}} \int dv\, \frac{\langle f \rangle}{(\omega - kv)} \left(\frac{v k_T}{v_{Th} k} - \omega + \omega \right)$$

$$= 1 + \frac{\omega_p^2}{(k v_{Th})^2} \int dv \langle f \rangle + \frac{\omega}{k} \frac{\omega_p^2}{(k v_{Th})^2} \int dv \frac{\langle f \rangle}{v - \frac{\omega}{k}}$$

$$= 1 + \frac{1}{k^2 \lambda_D^2} \left(1 + \frac{\omega}{k v_{Th}} \int d\varepsilon \frac{e^{-\varepsilon^2}}{\varepsilon - \omega / k v_{Th}} \right)$$

$$Z(\omega / k v_{Th}) = \int d\varepsilon\, e^{-\varepsilon^2} / \varepsilon - \omega / k$$

Plasma Dispersion Function
(Tabulated)